Classical Hair in String Theory II:

Explicit Calculations

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February 1, 2008

Abstract

After emphasizing the importance of obtaining a space-time understanding of black hole entropy, we further elaborate our program to identify the degrees of freedom of black holes with classical space-time degrees of freedom. The Cvetič-Youm dyonic black holes are discussed in some detail as an example. In this example hair degrees of freedom transforming as an effective string can be identified explicitly. We discuss issues concerning charge quantization, identification of winding, and tension renormalization that arise in counting the associated degrees of freedom. The possibility of other forms of hair in this example, and the prospects for making contact with D-brane ideas, are briefly considered.

1 Introduction: State-Counting, Deterministic Evolution, and Hair

The problem of understanding black holes as quantum-mechanical objects, although presumably academic for the forseeable future, has attracted great attention recently among theoretical physicists, due primarily to a fundamental conceptual tension.

The thermodynamic properties of macroscopic black holes are widely believed to be universal; that is, essentially independent of any details of the microscopic theory in which gravity is embedded. This belief is based on their remarkably simple derivation from very basic properties of the Einstein-Hilbert action [1, 2, 3, 4]. Indeed, the thermodynamic parameters of large black holes follow from general covariance, the causal structure of black hole backgrounds, and basic considerations of field theory at small curvature, which are plausibly independent of the matter content and other details of the theory.

On the other hand it is notorious that these thermodynamic properties lead to considerable conceptual tension with standard principles of quantum mechanics. In particular, the possibility that black holes formed under widely varying initial conditions will all eventually evaporate into approximately thermal radiation is not easily reconciled with the existence of a unitary S-matrix governing the dynamics. Indeed the evolution of many, possibly finely structured, initial states all into a common final state (or density matrix) – namely undistinguished thermal glop – could not follow consistently from fundamentally deterministic, time (or PCT) reversible equations for a closed system. This has led Hawking [5], and others [6], to suggest that no such

equations can exist, and that a new level of indeterminacy, beyond that of standard quantum mechanics, is inevitably associated with the existence of black holes.

The strength of these very broad arguments is open to serious doubt, however. The thermal character of the radiation was only established by approximate, semi-classical calculations for large black holes far from extremality. Given too naive a literal interpretation it is demonstrably false: in particular, the whole concept of thermal behavior becomes ambiguous for near-extremal holes [7, 8] and some specific quantitative corrections to thermality have been established [9, 10]. More generally, it is far from obvious that many approximately thermal states, with the same thermodynamic properties, cannot correspond to many different microscopically defined states differing in high-order, hard-to-calculate correlations: generically, of course, they do!

More profound, perhaps, are more detailed dynamical arguments tending toward the same conceptual tension. Without attempting to do these full justice, we can paraphrase their common core as follows [11, 12, 13, 14]. Since black holes are intrinsically structureless, or nearly so, well outside the horizon – "black holes have no hair" – the process of collapse followed by slow evaporation cannot be described by reversible, deterministic equations. For at intermediate stages – after the collapse and during the slow evaporation – the structureless black hole forms an extremely narrow information channel, which cannot accommodate a complete, accurate record of the initial state. For purposes of this argument it seems to be enough that there be no significant structure well outside the horizon, because that space-time region predominantly determines the asymptotic final state. Banks and O'Loughlin [15] especially forcefully and explicitly advocated that a distinc-

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tion must be made between states, measurable in the low energy theory, that should be counted towards the Bekenstein–Hawking entropy and other states, hidden from the observer strictly outside the hole, that should not.

Recently there have been some remarkable developments in the microscopic (quantum) theory of black holes in string theory. Early suggestions to identify certain electrically charged fundamental string states with extremal black holes, with vanishing classical area [16], led to a semi-quantitative matching of the number of microscopic states with the area of an appropriate stretched horizon [17, 18, 19] and their remarkably concrete realization [20, 21]. Subsequent suggestions and arguments [22] that a corresponding study of dyon states corresponding to extremal black holes with non-vanishing area [23, 24, 25] would enable a successful quantitative comparison with the classic Bekenstein-Hawking formula were confirmed [26] in a surprising and brilliant manner as a by-product of the identification of D-branes with string solitons [27] and development of an impressive associated calculational technology [28, 29, 30, 31, 32]. The understanding was immediately extended to near-extremal [33, 34] and general black holes [35, 36] although these results remain more controversial than the extremal case. Work along related lines has uncovered a beautiful, but still quite mysterious, relation between the quantization of 5–branes and the second quantization of strings [37, 38].

While these developments seem to us fundamentally to alter, perhaps permanently, the intellectual framework for the theoretical discussion, their limitations should not be underestimated. They apply at present only to very special, particular black holes in highly idealized models of physics. We shall discuss this sort of limitation, which may or may not be a purely technical problem, in more detail later. For present purposes, it is most important

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to notice that these developments primarily touch only the set of issues we discussed in paragraphs 2-4 above, and characterized as thermodynamic. The specifically D-brane methods of calculation are not couched in ordinary space-time terms, and do not immediately explain *why* a simple, universal result – namely, that of Bekenstein and Hawking – for the number of states should emerge. The dynamical issues, which may be the more profound ones, appear to require additional, or different, considerations.

The present work takes steps toward a microscopic evaluation of black hole entropy which emphasizes, rather than bypasses, space-time issues. Our working hypothesis, motivated by a desire to avoid the dynamical arguments noted above, is that the internal structure of black holes, and therefore ultimately their entropy the thermal properties, must be accurately represented by classical hair. We recognize that this is a radical hypothesis, and that it is not true in a generic effective field theory of gravity. After all, there are some powerful no-hair theorems which apply to simple, but widely used, effective field theories (e.g., the Standard Model) [39, 40, 41]. However, of course, the generic effective field theory of gravity almost certainly does not correspond to the reduction of an adequate quantum theory, as indicated by the presence of ultraviolet divergences. String theory is an increasingly popular candidate to provide an adequate quantum theory of gravity; and the effective field theories derived from it are much larger and more elaborate than have been commonly considered previously. The problem of insuring deterministic evolution may provide another, quite different indication that a substantial expansion of the dynamical arena is in order. Indeed, the loophole in the no-hair theorem we seek to exploit is intrinsically connected to the existence of additional dimensions. Our classical hair is related to the vast

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degeneracy among black holes that arises because the macroscopic charges of the black hole, which specify it uniquely to an observer who does not probe the compactified dimensions, do not uniquely specify the details of the configuration in the compactified dimensions. Although concrete identification of the complete catalogue of hair may be difficult in any particular case, this sort of construction appears to us sufficiently robust that it is a viable candidate for black hole entropy in general.

Conventional wisdom argues, in Kaluza-Klein reduction, that excitations with non–trivial dependence on small internal dimensions are very massive, so that only the constant mode is important in the low energy field theory. Although it invokes structure in the extra dimensions in a crucial way, our proposal does not contradict this conventional wisdom as it is usually applied. Local probes can only detect the hair if they are sensitive to the internal dimensions; since such probes must inevitably involve heavy degrees of freedom, there is a definite sense in which the hair is decoupled from the low energy world. However, if we are correct, these degrees of freedom do not decouple from the information flow in black hole processes, and must be kept if one is to obtain a closed (unitary) description.

It is a special virtue of this approach that the classical hair is measurable, in principle, at asymptotic distances. This is quite relevant for the information problem because hair stretching to infinity is retained permanently in boundary conditions, and is therefore in no danger of being 'lost' at intermediate stages. Thus the existence of abundant classical hair transforms this problem profoundly: retention of the information becomes the default option, which could only be endangered by unexpected strong coupling effects.

In the companion to this paper a general, rather abstract formalism em-

bodying our proposal was developed [4]. In the present paper we analyze a specific example, Cvetič–Youm dyon. The paper is organized as follows. In the first section we write the Cvetič–Youm dyon in its four-dimensional and ten dimensional forms and discuss the range of validity of these classical solutions. Next we turn to the explicit construction of classical hair. This was already outlined in the Appendix of [4] but here the solution is recast in canonical form and the effective surface theory identified explicitly. The semi-classical quantization of the black hole then follows. As a result we find that both the tension and the constraints on the effective string parametrizing the collective excitations of the black hole agree exactly with the analogous ones for a fundamental string. We then discuss discuss global issues that arise in counting classical hair states. Finally we discuss the relation of our program to other work on the subject and speculate briefly on the possibility of a more general and profound relation between classical hair and black hole entropy. The appendix is devoted to the quantization of the macroscopic black hole parameters.

2   The Cvetič–Youm dyon

Our working example is the Cvetič–Youm [23]. This is a black hole solution of the equations of motion that result from toroidal compactification of $N = 1$ supergravity in 10 dimensions. This theory in turn is the low-energy field theory limit string theory. Various properties of the solution were discussed in [24, 22]. In its full glory it is

$$
\begin{aligned}
ds^2 &= -\lambda dt^2 + \lambda^{-1} dr^2 + R^2(d\theta^2 + \sin^2\theta d\phi^2) \\
\lambda &= [(1 + \frac{\mathbf{Q^{(1)}}}{r})(1 + \frac{\mathbf{Q^{(2)}}}{r})(1 + \frac{\mathbf{P^{(1)}}}{r})(1 + \frac{\mathbf{P^{(2)}}}{r})]^{-\frac{1}{2}}
\end{aligned}
$$

$$R^2 = r^2[(1 + \frac{\mathbf{Q}^{(1)}}{r})(1 + \frac{\mathbf{Q}^{(2)}}{r})(1 + \frac{\mathbf{P}^{(1)}}{r})(1 + \frac{\mathbf{P}^{(2)}}{r})]^{\frac{1}{2}}$$

$$e^{2\phi} = \left[\frac{(1 + \frac{\mathbf{P}^{(1)}}{r})(1 + \frac{\mathbf{P}^{(2)}}{r})}{(1 + \frac{\mathbf{Q}^{(1)}}{r})(1 + \frac{\mathbf{Q}^{(2)}}{r})}\right]^{\frac{1}{2}} e^{2\phi_\infty}$$

$$G_{44} = \left(\frac{1 + \frac{\mathbf{P}^{(2)}}{r}}{1 + \frac{\mathbf{P}^{(1)}}{r}}\right) G_{44\infty} \;\; ; \;\; G_{99} = \left(\frac{1 + \frac{\mathbf{Q}^{(1)}}{r}}{1 + \frac{\mathbf{Q}^{(2)}}{r}}\right) G_{99\infty}$$

$$A_\phi^{(1)} = P^{(1)}(\pm 1 - \cos\theta) \;\; ; \;\; A_\phi^{(7)} = P^{(7)}(\pm 1 - \cos\theta)$$

$$A_t^{(6)} = -\frac{Q^{(6)}}{r + \mathbf{Q}^{(1)}} \;\; ; \;\; A_t^{(12)} = -\frac{Q^{(12)}}{r + \mathbf{Q}^{(2)}} \tag{1}$$

The black hole is parametrized by four independent charges, two electric and two magnetic. Their screened (boldfaced) incarnations have dimension of length, and are related to the physical charges through

$$(\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \mathbf{Q}^{(1)}, \mathbf{Q}^{(2)}) = \sqrt{\alpha'}(G_{44\infty}^{\frac{1}{2}} P_1, G_{44\infty}^{-\frac{1}{2}} P_7, G_{99\infty}^{\frac{1}{2}} Q_6, G_{99\infty}^{-\frac{1}{2}} Q_{12}) \tag{2}$$

In the remainder of the paper we take $G_{44\infty} = G_{99\infty} = 1$ for simplicity in notation. The appendix of this paper is devoted to a discussion of the normalization and quantization of the $U(1)$ charges.

The black hole is the 4 dimensional form of a solution to the underlying 10 dimensional theory. Using this correspondence (given precisely in the appendix eq. 37) the line element can be written in the obscure, but nevertheless convenient form

$$dS^2 = F du(dv + K du) + G_{ij}dx^i dx^j$$

$$G_{ij}dx^i dx^j = f[k(dx^{(4)} + \mathbf{P}^{(1)}(1 - \cos\theta)d\phi)^2 + k^{-1}(dr^2 + r^2(d\theta^2 + \sin\theta^2 d\phi^2))]$$

$$+ \sum_{i,j=5}^{8} \delta_{ij}dx^i dx^j \tag{3}$$

in string metric. Here $u = x^{(9)} - t$, $v = x^{(9)} + t$, and

$$F^{-1} = 1 + \frac{\mathbf{Q}^{(2)}}{r} \;\; ; \;\; K = \frac{\mathbf{Q}^{(1)}}{r} \;\; ; \;\; f = 1 + \frac{\mathbf{P}^{(2)}}{r} \;\; ; \;\; k^{-1} = 1 + \frac{\mathbf{P}^{(1)}}{r} \tag{4}$$

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The other nonvanishing fields are

$$B_{uv} = F \ \ ; \ \ \ B_{\phi 4} = \mathbf{P^{(2)}}(1 - \cos\theta) \ \ ; \ \ \ e^{\Phi} = Ff \tag{5}$$

This is the form of the black hole that appeared in eq. I.A.1[1]. It is repeated here for ease of reference.

We will work in the limit where string theory reduces to a 10 dimensional field theory. This is only reasonable if all scales are much larger than the string length $\sqrt{\alpha'}$. Specifically, it would be dubious to locate the horizon more precisely than the string length. However, physical distances ρ are related to coordinate distances by $\rho \simeq \mathbf{P^{(1)}}\mathbf{P^{(2)}}\ln r$ close to the horizon at $r = 0$; so this requirement concerns exponentially small coordinate distances and will not be relevant in this paper. It should be emphasized that this argument relies on the kinematics of extremal black holes only; so it is valid for a much wider class of black holes than the one considered here.

As explained in the introduction, black holes suffer from well-known conceptual problems that rely on one's having great confidence in the classical metric, even for their formulation. The persistence of trustworthy geometric data means that these problems still must be addressed.

3   The Classical Hair and the Constraints

Classical hair on the Cvetič–Youm dyon was exhibited already in the appendix of [4]. The purpose of this section, and the following, is to analyze classical hair using the canonical formalism developed in [4]. This provides a concrete example that complements the abstract formulae.

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[1]The roman numeral I refers to equation numbers in [4]. The A is a reference to the Appendix of that paper.

For the black hole without hair we choose the Schwarzchild time t as canonical time and split eqs. 3–5 in spatial and temporal parts, following [4]. The fields become

$$
\begin{aligned}
dS^2 &= g_{99}\,(dx^9)^2 + g_{ij}dx^i dx^j \\
g_{99} &= F(K+1) \\
N_9 &= -FK \\
N &= \sqrt{\frac{F}{K+1}} \\
B_{t9} &= -F
\end{aligned}
\tag{6}
$$

Here the transverse spatial metric g_{ij} is identical to the transverse spacetime metric G_{ij} in eq. 3. The additional non–zero fields ($B_{\phi 4}$ and e^Φ) will not be needed.

The canonical momenta can be calculated from eq. I.2.5 . The non–vanishing components are

$$
\begin{aligned}
\mathcal{E}^{r9} &= \frac{1}{2}\mathbf{Q}^{(2)} \\
\Pi_9^r &= \frac{1}{2}r^2\partial_r K
\end{aligned}
\tag{7}
$$

It is straightforward (but tedious) to verify explicitly that the fields in eqs. 6–7 solve the constraints eqs. I.2.8 – I.2.10 , as they should.

We are interested in a solution that is modified due to the presence of classical hair, and want to isolate the dynamics of the hair, following the strategy of sec. I.4 . Classical hair coordinates of the form identified previously are assumed, and their conjugate momenta follow from the constraints. For each canonical pair of background fields we can specify the field freely, but we must allow its conjugate momenta to respond to the presence of hair.

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Inspired by the backreaction in eq. I.A.11 , found using the Lagrange formalism, we will find it convenient that, for the g_{99} field, we specify instead $\Pi^{99} = 0$ and allow the field to depend on the hair. (This just amounts to a canonical redefinition of variables.) Now, the constraints eqs. I.2.8 – I.2.10 are rather complicated non–linear differential equations and we must solve for the momentum (and g_{99}) . The strategy will be to anticipate the momenta eq. 7 found in the absence of hair and introduce non–trivial components only when the constraints force it.

3.1   Gauge Hair

We will only consider gauge hair. The analogous calculation for Goldstone hair is in progress. In considerations later in the chapter we will assume, for purposes of discussion, that the result for gauge hair extends to other kinds of hair, though this is not at all obvious and may not be true.

From eq. I.A.9 we expect hair of the form

$$F_{rt} = -F_{r9} = \partial_r F \; q' = \frac{\mathbf{Q}^{(2)}}{(\mathbf{Q}^{(2)} + r)^2} \; q' \tag{8}$$

We suppress the internal index $I = 1, \cdots, 16$. In the canonical formalism the function q' is specified as a function of x^9 at a fixed time slice; so the time dependence will only appear when the equations of motion are considered in sec. 4.2. The prime denotes derivative with respect to x^9, and we should note that q is normalized differently than in [22]. We choose the gauge $A_9 = A_t = 0$ and find

$$A_r = \partial_r F \; q \tag{9}$$

The momentum $\mathcal{E}^9$ is non–dynamical because it is conjugate to $A_9 = 0$. It

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can be verified by writing out eq. I.2.9 that it enters the Hamiltonian as

$$H = \frac{1}{2} N g^{-\frac{1}{2}} e^{2\Phi} g_{99} (\mathcal{E}^9 + \mathcal{E}^{9r} A_r)(\mathcal{E}^9 + \mathcal{E}^{9r} A_r) + \text{terms independent of } \mathcal{E}^9 \quad (10)$$

We can impose the corresponding equation of motion as a rigid (first class) constraint by making the identification

$$\mathcal{E}^9 = -\mathcal{E}^{9r} A_r \quad (11)$$

in the following. Noting that $\mathcal{E}^{r9}$ is a constant, antisymmetric in the two indices, the Gauss' law constraint eq. I.2.8

$$\partial_\alpha (\mathcal{E}^\alpha - \mathcal{E}^{\alpha\beta} A_\beta) = \partial_9 (\mathcal{E}^9 - \mathcal{E}^{9r} A_r) + \partial_r (\mathcal{E}^r - \mathcal{E}^{r9} A_9) = 0 \quad (12)$$

can be integrated to

$$\mathcal{E}^r = -2\mathcal{E}^{r9} F \; q' = -\frac{\mathbf{Q}^{(2)}}{(1 + \frac{\mathbf{Q}^{(2)}}{r})} q' \quad (13)$$

A constant of integration was undetermined *a priori*. With the choice eq. 13 the momentum agrees with the on–shell result that can be obtained from eq. I.2.5 .

Now consider the Hamiltonian constraint eq. I.2.9. The gravitational momenta and the metric function K are unknowns. Expanding the curvature symbol and writing out the electromagnetic terms we find

$$\mathcal{H} = g^{-\frac{1}{2}} e^{2\Phi} [\text{Tr } \Pi^2 + \frac{1}{2} \Pi^\Phi \text{Tr } \Pi + \frac{D-2}{16} \Pi_\Phi^2] - -g^{\frac{1}{2}} e^{-2\Phi} g^{rr} \frac{1}{2(K+1)^2} (\partial_r K)^2$$

$$+ \; g^{\frac{1}{2}} e^{-2\Phi} g^{rr} \frac{1}{K+1} [\frac{1}{F} (A_r')^2 + \frac{1}{r^2} \partial_r (r^2 \partial_r K)] = 0 \quad (14)$$

after a lengthy calculation. Requiring the last parenthesis to vanish by itself gives K in terms of the hair

$$K = \frac{\mathbf{Q}^{(1)}}{r} - \frac{\mathbf{Q}^{(2)2}}{2r(r + \mathbf{Q}^{(2)})} (q')^2 \quad (15)$$

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The constant at infinity was determined by the prescribed charge of the black hole. This expression agrees with eq. I.A.11 . The other terms in eq. 14 will be considered below.

Next we consider the supermomentum constraints $\mathcal{H}_\alpha = 0$ arising from eq. I.2.10 . Recall that the gravitational momenta are tensor densities so that

$$\Pi^{\alpha\beta}{}_{|\beta} = \Pi^{\alpha\beta}{}_{,\beta} + \Gamma^\alpha{}_{\beta\gamma}\Pi^{\beta\gamma} \tag{16}$$

Assuming that only $\Pi^r_9 \neq 0$ the $\mathcal{H}^9 = 0$ constraint becomes

$$\Pi^{9\beta}{}_{|\beta} = g^{99}\partial_r(g_{99}\Pi^{9r}) = \frac{1}{2}F^{9\alpha}(\mathcal{E}_\alpha + \mathcal{E}_{\alpha\beta}A^\beta) = \frac{1}{2}g^{99}\partial_9 A_r\, \mathcal{E}^r \tag{17}$$

and the momentum is found to be

$$\Pi^r_9 = -\frac{\mathbf{Q}^{(2)}}{4(1 + \frac{\mathbf{Q}^{(2)}}{r})^2}(q')^2 + \text{constant} = \frac{1}{2}r^2\partial_r K \tag{18}$$

The constant of integration is independent of r but could depend on x^9. It was determined so that the first line in eq. 14 vanishes without forcing additional non–zero momenta . The formula for Π^r_9 eq. 18 is formally identical to eq. 7, which is valid when there is no hair, but the expression eq. 15 for K now depends on the hair.

With a certain exception, discussed in sec. 3.2, the remaining supermomentum constraints $\mathcal{H}_\alpha = 0$ are not affected by the gauge hair.

3.2 The Final Constraint

If only the hair variables depended on x^9 we would at this point have checked all the constraints. However the backreaction on the geometry is taken into

account by allowing g_{99} to depend on the hair variables, and non–trivial dependence of the metric on x^9 ensues. By inspecting the appropriate expansion of the Ricci scalar one can show that the Hamiltonian constraint remain satisfied if this dependence is allowed. However, since Π^9_r in eq. 18 depends implicitly on the hair, the supermomentum constraint $\mathcal{H}^r = 0$ require further consideration. The *ansatz*

$$\Pi_{ij} = -\frac{1}{4} \ g_{ij} \ \Pi^\Phi = \frac{1}{D-2} \ g_{ij} \mathrm{Tr} \ \Pi \tag{19}$$

can be interpreted geometrically as a non–zero extrinsic curvature $\mathrm{Tr} \ \mathbf{K}$, using eq. I.2.5 . It is an easy but non–trivial calculation to verify that the Hamiltonian constraint remains satisfied. Using the *ansatz* and the explicit form of the metric we find

$$\Pi^{rj}{}_{|j} - \frac{1}{2}\Pi^\Phi \partial^r \Phi = \frac{r^2}{F}\partial_r[\frac{F}{r^2}\frac{1}{D-2}\mathrm{Tr} \ \Pi] = \Pi^{r9}{}_{|9} \tag{20}$$

with the last equality expressing $\mathcal{H}^r = 0$. Using eq. 15 and the metric this can be integrated to

$$\frac{1}{D-2}\mathrm{Tr} \ \Pi = -r^2 fk^{-1}\frac{1}{2F(K+1)}\partial_9 K \tag{21}$$

We choose the constant (that may depend on x^9) to vanish.

The appearance of $r^2 fk^{-1} = (r + \mathbf{P^{(1)}})(r + \mathbf{P^{(2)}})$ is a new and tantalizing feature. In all our previous calculations the magnetic charges canceled in final results. $\mathrm{Tr} \ \Pi$ vanishes on–shell, but the off-shell value can be important for the quantization of the system. In fact, as we shall see in sec. 5, the tension renormalization anticipated in [22] can be interpreted as failure of single valuedness in the x^9 variable. This phenomenon may be signaled by total derivative terms in the Hamiltonian density, and this is indeed how $\mathrm{Tr} \ \Pi$ enters, as alluded to in the end of sec. I.5 . A precise connection eludes us at this point, however, and in sec. 5 a different reasoning will be pursued.

4 The Effective Surface Theory and Quantization

In [4] properties of the effective theory for the classical hair were derived in abstract form. Using the expressions from the previous section they can now be made explicit.

It was proposed in [4] that the condition

$$\Pi_\alpha^r = 0 \quad ; \quad \alpha \neq r \tag{22}$$

should be imposed at $r = 0$. The only non–trivial condition arises for $\alpha = 9$ and it implements reparametrization invariance in the 9th direction. Using eq. 18 and eq. 15 we find

$$\frac{1}{2} \mathbf{Q}^{(2)} \langle (q')^2 \rangle = \mathbf{Q}^{(1)} \tag{23}$$

This relation has a profound consequence: it forces the existence of hair. The physical interpretation is that the hair must carry all the momentum (*i.e.*. charge, after compactification) that is seen at infinity. The analogous condition for the fundamental string was derived in [20] by appealing to a cosmic censorship hypothesis, and in [21] by insisting that the external field matches on to a string source. It is interesting that our requirement of no source agrees with the result that follows from specific properties of a string source.

4.1 Quantization of Gauge Hair

In [20, 21] the condition eq. 23 was identified with the matching condition on the world sheet of a string that acts as a source. Here we have no source

and proceed instead to quantize the field directly. In the complete theory the Poisson brackets of the fields are

$$\{\mathcal{E}^r(\vec{x}), A_r(\vec{x}')\} = 16\pi G_N \,\frac{1}{2}\delta(\vec{x} - \vec{x}') \tag{24}$$

(The normalization of the right hand side arises because we have defined momenta as variations of $16\pi G_N L$.) This induces brackets in the reduced theory.

The momentum conjugate to the hair variables is essentially the x^9 derivative of the hair, as is characteristic of chiral fields. The factor of $\frac{1}{2}$ on the right hand side of eq. 24 is the notorious one that arises in the quantization of chiral fields when the second class constraints are properly implemented.

Using the explicit expressions eq. 9 and eq. 13 we perform the radial integral and find

$$4\pi\{-\frac{1}{2}\mathbf{Q}^{(2)}q'(x^9), q(x'^9)\} = 16\pi G_N \,\frac{1}{2}\, L\delta(x^9 - x'^9) \tag{25}$$

The trivial angular integral gave rise to the 4π on the left hand side. The explicit volume factor $L = 2\pi\sqrt{\alpha'}$ was implied in the normalization of the δ–function in x^9 and is necessary for G_N to denote the four–dimensional Newton's constant throughout.

We introduce Fourier modes

$$q(x^9) = \sum_n q_n \, e^{\frac{2\pi i n}{L}x^9} \tag{26}$$

and find

$$\frac{1}{4G_N}\{-\frac{1}{2}\mathbf{Q}^{(2)}\frac{in}{\sqrt{\alpha'}}q_n, q_{n'}\} = \frac{1}{2}\delta_{n+n'} \tag{27}$$

Replacing the brackets with commutators, and using eq. 26 again, we find

$$\frac{\sqrt{\alpha'}}{4G_N}\frac{1}{2}\mathbf{Q}^{(2)}\,\langle\,(q')^2\,\rangle = N \tag{28}$$

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Here the level is $N = \sum_n n N_n$ in terms of occupation numbers of individual states N_n. Now the matching condition eq. 23 can be written

$$N = \frac{\sqrt{\alpha'}}{4 G_N} \mathbf{Q}^{(1)} = \frac{2 Q^{(1)}}{g_{\text{st}}^2} = q^{(1)} \tag{29}$$

in terms of the quantized charges of the Appendix. Both sides of this relation take all integer values. It should be noted that, when expressed in terms of integers, the matching condition is independent of the proportionality constant relating G_N and $\alpha' g_{\text{st}}^2$ despite the ubiquity of the parameters at intermediate stages.

The normalization in eq. 29 agrees with the standard one obtained from world sheet considerations. Its physical origin is the requirement that the oscillators carry the total momentum is also related. The nonrenormalization of string tension discovered by Dabholkar and Harvey equates the tension of the fundamental string with the tension that is measurable at infinity [42]. Here we go a step further and treat the oscillations as *bona fide* collective excitations which are quantized without any reference to the microscopic theory.

4.2   Equations of motion

The reduced Hamiltonian eq. I.5.2 has the simple form

$$16\pi G_N H_{\text{red}} = 2 \int_{r=0} \Pi_9^r = - \int_{r=0} [\mathbf{Q}^{(1)} - \frac{1}{2} \mathbf{Q}^{(2)} (q')^2] \tag{30}$$

The equations of motion are found by variation of the Hamiltonian. The relation between the variables q' that appear in this expression and the canonical variables appropriate for the variational principle were worked out in the pre-

vious sections. We find the simple expression

$$H_{\text{red}} = -\int dx^9 \; q' \, \pi_q \tag{31}$$

where we denoted by π_q the canonically conjugate to q and omitted a constant. The equations of motion become

$$\frac{\delta H_{\text{red}}}{\delta \pi_q} = -q' = \dot{q} \;\; ; \quad \frac{\delta H_{\text{red}}}{\delta q} = \pi_q' = -\dot{\pi}_q \tag{32}$$

where the dots denote time derivatives. The equations are easily solved and a time dependence is found that demands q, as well as π_q, to be functions of $x^9 - t$. We conclude that only left movers are solutions of the equations of motion, as anticipated.

The strategy is more important than the result: the hair variables were isolated from the background, in a canonical and off-shell fashion, before the equations of motion were found and solved. It was presumably special properties of the example here that allowed simple, explicit expressions to be found; but the calculation serves as a useful paradigm for the more general case, discussed abstractly in [4].

4.3 No On-shell Area

In sec. I.4 we gave a simple and general derivation of the black hole entropy from the macroscopic view point. It was the second term in the reduced Hamiltonian eq. I.2.10 the was responsible for entropy in the general setting. In the current explicit example it has the form

$$H_{\text{red}} = -\frac{\pi}{G_N}\Theta\sqrt{\mathbf{P}^{(1)}\mathbf{P}^{(2)}\mathbf{Q}^{(2)}[\mathbf{Q}^{(1)} - \frac{1}{2}\mathbf{Q}^{(2)}(q')^2]} \tag{33}$$

It was omitted above because the temperature $\Theta = 0$ for the extremal black hole under consideration. However, the form of this term makes manifest

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some properties of the formalism that are presumably generic. First of all it emphasizes the feature that the area is dynamical; so within the reduced Hamiltonian formalism this term, and specifically the temperature, is expected to enter the equations of motion explicitly.

It is intriguing that the term in the reduced Hamiltonian which, in the absence of hair, quite explicitly can be interpreted as the entropy is dramatically modified in the presence of hair. Indeed, if we impose the matching condition eq. 23, the dynamical area of the black hole vanishes! This is gratifying, because if the area were non–zero the black hole with hair – supposedly a microscopically specified state – would itself be a thermodynamic body with internal structure [2]. This motivation appears so compelling, that after having observed the phenomenon in a specific example, we are led to expect a vanishing area for the general microstate of any black hole. We should emphasize in this connection that it is the modulus g_{99} that shrinks close to the horizon, with the spatial geometry in the uncompactified dimensions exhibiting no conspicuous features.

It is interesting to speculate on possible dynamical consequences [3]. Consider an infalling observer who may or may not be translationally invariant in the x^9 direction. In a given microstate the black hole has structure in the x^9 direction, so the impinging observer encounters a potential, because the equations are non-linear, and will generically develop structure too. Of course whenever an incoming wave experiences a non–trivial potential, there is a reflected wave, and as the modulus g_{99} shrinks to zero the relevant effective potential appears to become arbitrarily strong, and reflection complete. This

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[2]This was pointed out to us by A. Sen.

[3]This paragraph was the result of a discussion with S. Mathur.

suggests a mechanism — realizable within the semiclassical approximation — that avoids information loss in a very real sense: it becomes impossible to fall into the black hole. This phenomenon, an infinitely high effective barrier that reflects incoming waves, was previously encountered for a certain special class of black holes by Holzhey and Wilczek [43].

5 Tension Renormalization

A satisfactory understanding of the internal structure of black holes must include a quantitative agreement between the number of microstates and the entropy that follows from macroscopic considerations. Using the statistical mechanics of ideal gasses in two dimensions, the matching condition eq.29, can be translated into an entropy

$$S \simeq 4\pi\sqrt{q^{(1)}} \tag{34}$$

This microscopic result is much smaller than the macroscopic one from the appendix eq. 47, which contains the product of all four charges (two electric, two magnetic). The discrepancy lead Horowitz and Marolf to conclude that the classical hair form mesoscopic structure, unrelated to the true microstructure that account for the black hole entropy [44, 45]. Our interpretation is quite the contrary: the qualitative picture is viable but the counting is incomplete because global issues have been slighted.

5.1 Winding Strings

For a first indication of this possibility note that the matching condition eq. 29 differs from the standard perturbative matching condition by a factor

of $q^{(2)}$. This discrepancy derives directly from the canonical brackets eq. 24. The arguments of the fields and the δ–function on the right hand side include x^9, but in string theory the spacetime coordinate is related to the world sheet coordinate σ by a multiplicative integer winding number w, *i.e.* $x^9 = w\sigma$. The winding number couples to the Kalb–Ramond field, characterized by the quantum $q^{(2)}$; so we are led to identify w with $q^{(2)}$ and use the variable $\sigma = \frac{1}{q^{(2)}}x^9$ in the canonical brackets and the Fourier transform. Then the perturbative matching condition discussed above, *viz.*

$$N = q^{(1)}q^{(2)} \tag{35}$$

is then neatly recovered.

Considering the physical nature of a string it is perhaps unsurprising that there is a winding number. However, an observer of the field created by a string will detect momenta that are smaller than is normally allowed in a volume with the dimension of the compactified space. These small momenta are quite puzzling, but certainly real and measurable, even at infinity. Their physical manifestation is the failure of recovering measured values of the fields upon circumnavigating the compactified dimension. To parametrize this, the observer may adopt a new definition of the dimension of the compactified space ($L \to q^{(2)}L$) or, alternatively, conclude that in this environment the string tension is larger than its universal value $T = \frac{1}{2\pi\alpha'}$. The phenomenon of small momenta is referred to as tension renormalization [17, 22, 46, 47]. The concept has recently been made mathematically precise [48] with a relation to the winding sectors of the orbifold appearing in [35].

As an aside, consider a component of the field with momentum that is larger than the minimal value (by a factor that is not a prime). Contributions to the amplitude arise from individual, maximally wound strings

with this momentum; but also from multiple string configurations that each wind less. In the underlying Hilbert space of string field theory a distinction should presumably be made between such contributions, and this may affect the proper counting of degeneracy. However, it would seem very difficult for the observer far away to discern such subtle differences, even though careful experiments using Aharonov–Bohm type correlations might conceivably accomplish it. Moreover, since the configurations in question have identical classical actions it is possible that, even in principle, they should not be distinguished when calculating the volume of classical phase space that should be compared to the Bekenstein–Hawking entropy. Fortunately the leading order result for the entropy does not seem to depend on this issue, and we can simply assume that a single effective string suffices.

5.2   Magnetic Flux

At this point we have recovered the well-known degeneracy of the fundamental string from a spacetime point of view. In other words we have found a non–renormalization result for the tension that expresses the absence of gravitational backreaction on the classical hair. It seems, alas, that the classical hair considered here is simply the external field of a fundamental string propagating in an inert black hole background, rather than an effective string theory that parametrize collective excitations of the black hole. However, as it currently stands the counting gives a result that is inconsistent with duality, a symmetry of the low energy theory. This symmetry interchanges electric and magnetic charges, but leaves the degeneracy of states (*i.e.* here the entropy) invariant. So we know the understanding must be incomplete.

First reconsider how the winding emerges technically: it amounts to the

appearance of a phase in the hair variables when they are taken around the x^9 direction. Then it can be concluded that the Fourier transform eq. 26 uses a length L which is too small, because only when the effective length is increased do proper single valued fields arise.

The possibility of this kind of phase can be understood by reasoning along the lines of sec. 2 , as follows. In a background of charge $q^{(2)}$, gauge transformations with gauge function Λ of minimal period 2π change the action by $2\pi q^{(2)}$. A transformation that formally acts like a gauge transformation with gauge function Λ of period $2\pi/q^{(2)}$ generates new solutions that are *not* gauge equivalent to the original one. The field theory of classical hair allows Wilson lines created by this kind of transformation to be attached to each hair independently. The mechanism proposed here to generate more classical hair, once we have discovered some, is closely related to the construction of discrete gauge hair in [49]. All winding modes can be accounted for this way.

We now turn our attention to the role of the magnetic charges. Again the background allows 'small' gauge transformations that generate new solutions from existing ones. From the requirement that the Dirac–string remains hidden the allowed periodicities are $\frac{2\pi}{p^{(1)}}$ (or $\frac{2\pi}{p^{(2)}}$ for the other gauge field). Again we can apply this kind of transformation on each of the modes of the classical hair independently. In consequence, configurations exist that, as we move around the 9th direction, change by a phase. If $p^{(1)}$, $p^{(2)}$, and $q^{(2)}$ are relatively prime the effective length, needed to obtain a single-valued field, becomes the product $q^{(2)}\ p^{(1)}\ p^{(2)}$. With this tension renormalization the correct matching condition entropy are recovered.

To put this heuristic argument on a firmer basis it must be verified more explicitly that the appropriate couplings are present, for all modes of the

classical hair, and a better understanding of the field theoretic basis for independent transformations on each mode is also needed. Though much remains unsettled, we do think it is now quite plausible that the right sort of field theory contains the features necessary for a quantitative account of the black hole entropy using classical hair, once global issues are properly addressed.

5.3   Relation to Other Work

Cvetič and Tseytlin [24] (see also [50, 51]) noted that the classical hair can be represented as marginal deformations in the underlying conformal field theory. In this approach the appropriate tension renormalization appears naturally with the predicted coefficient, as the Kac–Moody level of the current algebra.

The left hand side of the matching condition is the level of $c = 24$ physical oscillators of heterotic string theory. As explained in the Appendix the formula for the black hole entropy depends on the spectrum of allowed magnetic charges. If Dirac quantization and a matching condition with minimal charges are used, as seems most reasonable, there is a discrepancy of a factor of 2 between the renormalized version of eq. 34 and the Bekenstein-Hawking result eq. 47. Instead $c = 6$ is required to bring the renormalized the results into agreement.

Tseytlin noted that $c = 6$ could arise naturally from classical hair of the type presented here if only the Goldstone hair in the $(r, \theta, \phi, 4)$ directions suffer tension renormalization [51]. (The contribution of fermionic partners presumably follows automatically if spacetime supersymmetry is realized, although we have not considered this carefully within our framework.) Un-

fortunately the modes in the $(r, \theta, \phi, 4)$ are precisely the ones we have not been able to handle analytically.

Strominger and Vafa presented a calculation in the type II theory that represents the microscopic states of certain 5 dimensional black holes as an effective string that indeed has $c = 6$ [26]. The effective string that appears in this accounting for black hole entropy is not a renormalized version of the fundamental string, but rather some new kind of chiral string that has $c = 6$, and is confined to live on 6–dimensional world volumes. It was proposed by Dijkgraaf, Verlinde, and Verlinde that this string should be taken as a serious starting point for quantization [52, 37].

Translating this reasoning into our spacetime picture we are lead to expect that, in addition to the classical hair we have exhibited explicitly, there are some excitations that depend on the (x^5, x^6, x^7, x^8) variables as well as $x^9 - t$ and that only those species suffer tension renormalization. Such classical solutions would be consistent with supersymmetry and therefore quite plausibly exist, but their explicit construction may be difficult.

6   Discussion

At the core of our program, to account for black hole entropy using classical hair, is a connection between global constraints a field configuration – here, that it contains a black hole with given macroscopic quantum numbers – and some characterization of the number of solutions which realize it. This is a theme that has occurred before in physics and mathematics, as the connection between anomalies and zero-modes, or as index theorems [53]. Here we seek an index theorem with a new element, in that the number

of solutions is quantified as a volume – the volume of classical phase-space. If such an index theorem were found, it would for many purposes free us from the necessity of actually finding the explicit solutions, which as we have seen is a painful process at best. Ideally, it would relate an 'anomaly' in information flow across the horizon to the build-up of classical hair outside, and thus could be interpreted as a necessary physical consistency condition on the effective theory. At present, however, these ideas are no more than attractive speculation.

Our concept of classical hair that is measurable far away implies that all information is contained in scattering amplitudes that are susceptible only to the leading behavior of the potentials, *i.e.* small angle scattering. This appears closely related to the program pursued by Mandal and collaborators [54, 55] and to the calculations of Das and Mathur [46, 56]. On the other hand we see no obvious connection with the statistical hair advocated by Strominger [57] or the quantum hair discussed by Banks [58].

't Hooft has advocated for some time that the consistency of black hole quantum mechanics severely constrains the form of the underlying microscopic theory [59]. We hope to have made it plausible that black hole entropy can be accounted for microscopically, but only in special many-dimensional theories with appropriate symmetries. This seems to embody a part of 't Hooft's program.

discussion.

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A Quantization of Macroscopic Charges

This Appendix contains an elementary discussion, within the framework of field theory, of the quantization conditions on the $U(1)$ charges.

Consider dilaton gravity in ten dimensions, coupled to an antisymmetric vector field. After toroidal compactification to 4 dimensions the Lagrangian becomes

$$
\begin{aligned}
L \;=\; & \frac{1}{16\pi G_N} \int d^4x \sqrt{-g}[R_g - 2\partial^\mu\phi\partial_\mu\phi \\
& - \frac{\alpha'}{4} e^{-2(\phi-\phi_\infty)} \sum_{a=1}^{6} (F^{(a)}_{\mu\nu} G^{aa} F^{(a)\mu\nu} + F^{(a+6)}_{\mu\nu} G_{aa} F^{(a+6)\mu\nu}) \\
& + \frac{1}{4}\partial^\mu G^{ab}\,\partial_\mu G_{ab}]
\end{aligned}
\tag{36}
$$

The fields $F^{(a)}_{\mu\nu}$ ($a = 1,\cdots,6$) are Kaluza Klein fields, i.e. dimensionally reduced components of the metric field, and the fields $F^{(a+6)}_{\mu\nu}$ ($a = 1,\cdots,6$) are winding fields that arise from the antisymmetric vector field. The compactification scale $\sqrt{\alpha'}$ sets the relative scale of the terms. We have ignored possible moduli from compactification of B_{IJ} and $A_I^{(i)}$; so this toy model contains only the most basic features of string induced gravity. For the heterotic string the full bosonic Lagrangian is given in [60].

In the toroidal compactification that leads to the Lagrangian eq. 36 the Kaluza–Klein fields are introduced by the decomposition

$$
dS^2 = G_{\mu\nu}dx^\mu dx^\nu + G_{mn}(dx^m + A_\mu^{(m)}\sqrt{\alpha'}\,dx^\mu)(dx^n + A_\nu^{(n)}\sqrt{\alpha'}\,dx^\nu) \tag{37}
$$

of the string metric $G_{IJ} = e^{2(\phi-\phi_\infty)}g_{IJ}$. Therefore the gauge invariance $A_\mu \to A_\mu + \partial_\mu\Lambda$ (for each of the gauge fields) is induced by reparametrization invariance of the appropriate internal dimension. The gauge function is only

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defined up to periodicity because the internal coordinate is periodic. Having adopted the convention that the moduli at infinity are unity the radii of the tori are simply $\sqrt{\alpha'}$ in the current conventions and the appropriate identification becomes $\Lambda \equiv \Lambda + 2\pi$.

Consider a single Kaluza–Klein field

$$L = \frac{1}{8\pi g_{\text{st}}^2} \int e^{-2(\phi - \phi_\infty)} F_{\mu\nu} F^{\mu\nu} \tag{38}$$

It is normalized as in eq. 36 if we use $G_N = \frac{1}{8}\alpha' g_{\text{st}}^2$ as a convention that defines the string coupling. The action has a subtle gauge dependence that is due to boundary terms at infinity. Indeed, for solutions to the equations of motion,

$$\delta L = \frac{1}{2\pi g_{\text{st}}^2} \int_{r\to\infty} F_{\mu\nu} \delta A^\nu dS^\mu = \frac{2Q}{g_{\text{st}}^2} \int \delta A_t dt \tag{39}$$

where the charge was defined by the asymptotic behavior $F_{rt} \to \frac{Q}{r^2}$ as $r \to \infty$.

As variation in eq. 39 we consider the pure gauge transformation $\delta A_t = \partial_t \Lambda$. Recalling that the Λ admits periodic identifications and imposing the quantization condition that the Lagrangian similarly is well defined up to multiples of 2π, we find the quantization condition

$$\frac{2Q}{g_{\text{st}}^2} = \text{integer} \equiv q \tag{40}$$

This holds for each Kaluza–Klein field separately.

For gauge fields that derive from dimensional reduction of the Kalb–Ramond field B_{IJ} the situation is less straightforward. T–duality indicates that such charges must have the same quantization rule, and this will be adhered to in the following. For a heuristic understanding of how this emerges in a spacetime approach, implement T–duality by large coordinate transformations that interchange time with one of the internal coordinates. On the

world sheet this symmetry is related to modular invariance; and in field theory it presumably follows analogously from general covariance and unitarity. The effective periodicity of the Euclidian time plays the role of temperature; so it is intriguing that it appears in the quantization conditions.

The Dirac quantization condition on the magnetic charges is also a consequence of the compactification: the magnetic gauge potentials are $A_\phi = P(\pm 1 - \cos\theta)$ on the north and south hemisphere, respectively . On the equator they are related by $A_\phi = A'_\phi + \partial_\phi \Lambda$; so $\Lambda = 2P\phi$ must be an acceptable transition function. The implied periodicities from this and from the compactification are $\Lambda \equiv \Lambda + 4\pi P \equiv \Lambda + 2\pi n$, so

$$2P = \text{integer} \tag{41}$$

is the Dirac quantization condition for the magnetic charge.

In field theory there is no fundamental requirement that all integer magnetic charges are realized in a given theory. However, in recent years evidence has been accumulating that string theory indeed saturates Dirac's quantization condition. Perhaps the most convincing argument appears in type I string theory where solitonic objects have explicit realizations as D–branes; and T–duality ensures that any number of D–branes must be realized. In this context it can be verified by direct calculation [27] (or from topological considerations [61]) that the D–branes indeed carry the minimum Dirac quantum. The corresponding result in heterotic string theory then follows as a prediction of duality.

The quantization on magnetic charges can be understood from duality in the field theory limit, as follows [62]. Impressive evidence for non–abelian duality has accumulated for $N = 4$ supersymmetric gauge field theory (for a review see [63]). Normalizing the $SU(2)$ gauge field part of the Lagrangian

as

$$L = \frac{1}{16\pi} \int \left[\frac{4\pi}{e^2}\text{tr } W_{\mu\nu}W^{\mu\nu} + \frac{\theta}{2\pi}\text{tr } \tilde{W}_{\mu\nu}W^{\mu\nu}\right] \tag{42}$$

where

$$W_{\mu\nu}^{(a)} = \partial_\mu A_\nu^{(a)} - \partial_\nu A_\mu^{(a)} + \epsilon_{abc}A_\mu^{(b)}A_\nu^{(c)} \tag{43}$$

the non–abelian duality is implemented by the $SL(2,Z)$ parameter

$$\tau = \frac{\theta}{2\pi} + \frac{4\pi i}{e^2} \tag{44}$$

For $\theta = 0$ the Z_2 subgroup $\frac{4\pi}{e^2} \leftrightarrow \frac{e^2}{4\pi}$ is a special case of the larger duality group. The spectrum of the theory includes dyons with electric charges quantized so that $\frac{4\pi}{e^2}Q$ is an integer and magnetic charges quantized as integers. The spectrum satisfies Dirac's quantization condition with the magnetic quanta equal to twice their minimal values. This apparent doubling arises because both the electric and the magnetic charges are in the adjoint representation of the gauge group, rather than in the fundamental.

To find the quantization condition on heterotic string theory note that it develops non–abelian gauge symmetry at special points in the moduli space. At such points the $U(1)$'s in eq. 36 can be considered subgroups of $SU(2)$ gauge groups and the quantization conditions inferred from the non-abelian field theory. The result is extended to the rest of moduli space, by continuity.

Concretely, consider a Kaluza–Klein field $F^{(1)}$ and a winding field $F^{(2)}$ and form $F^{(\pm)} = F^{(1)} \pm F^{(2)}$. The $U(1)$ field $F^{(-)}$ is part of a non-abelian gauge field and we identify it with $W^{(3)}$. Now, two copies of the Lagrangian eq. 38 become

$$L = \frac{1}{16\pi g_{\text{st}}^2} \int e^{-2(\phi-\phi_\infty)} \left[F^{(-)2} + F^{(+)2}\right] \tag{45}$$

38

Regarding $F^{(+)}$ a fixed background field and identifying g_{st}^2 with $\frac{e^2}{4\pi}$ this agrees with an abelian subgroup of the Lagrangian eq. 42 for constant dilaton. The electric (and magnetic) charges of $F^{(1,2)}$, $\frac{1}{g_{st}^2}Q^{(1,2)}$ (and $P^{(1,2)}$) are quantized in half integers by eq. 40 (and eq. 41) so, keeping the charges of $F^{(+)}$ fixed at an even integer, the allowed electric (and magnetic) charges of $F^{(-)}$ are all the integers, in accordance with the non–abelian field theory[4]. The string theory duality (with the minimum Dirac quantum) therefore maps correctly to the non–abelian field theory duality (with twice the minimum quantum). The precise form of the embedding plays a crucial role in making the factors of two agree.

As an application of the quantization rules extract the ADM mass from the metric in eq. 1 and apply the quantization rules on each charge, to find

$$M_{\text{ADM}} = \frac{\sqrt{\alpha'}}{4G_N}[Q^{(1)}+Q^{(2)}+P^{(1)}+P^{(2)}] = \frac{1}{\sqrt{8G_N}}[g_{st}(q^{(1)}+q^{(2)})+\frac{1}{g_{st}}(p^{(1)}+p^{(2)})] \tag{46}$$

Note that G_N entered through boundary conditions at infinity that define the ADM mass. This formula has several applications. The invariance under inversion of the coupling constant and simultaneous interchange of electric and magnetic quantum numbers can be taken as evidence for duality. Moreover, for states with no magnetic charge it agrees with the standard world sheet expression if the integers q^1 and q^2 are identified with the integer world sheet momentum and winding respectively. In either case the consistency checks are independent of the proportionality constant relating G_N and $\alpha' g_{st}^2$, and therefore of the quantization condition.

The quantity of primary concern to us is the thermodynamic entropy of

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[4]An odd $F^{(+)}$ charge introduces a shift in the allowed $F^{(+)}$ charges by one half but the spacing remains the same.

the black hole. It is

$$S = \frac{A}{4G_N} = \frac{1}{4G_N} 4\pi\alpha'\sqrt{Q^{(1)}Q^{(2)}P^{(1)}P^{(2)}} = 2\pi\sqrt{q^{(1)}q^{(2)}p^{(1)}p^{(2)}} \qquad (47)$$

Again G_N entered through boundary conditions at infinity. As discussed in [22, 64], the definition of the string coupling dropped out of this expression as did the moduli (which are set to unity in the conventions here). It therefore relies on the quantization condition on the magnetic charges only.